Analysis of Stability Radius of Inverted Pendulum on a Cart System

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Abstract:

This paper discusses the concept of stability radius and presents an evaluation of the same for inverted pendulum on a cart system. In a system, robustness can be calculated or determined by using stability radius. It is shown through example that the usual margins (gain and phase margins) may not always present an accurate measure of stability robustness. Simulation results indicate that some parameters in the system may affect the stability radius more than the others. The results also indicate the effect of simultaneous changes in system parameters on the stability radius may be highly nonlinear and non-monotonic. Finally, a design guideline for inverted pendulum mass to length ratio has been derived from the results.

Keywords:

Stability Radius, Parameter Variation Analysis, Inverted Pendulum on a Cart

1. Introduction

Stability is one of the major concerns in control systems. A good amount of research has been done on achieving stability in multiple systems. In nonlinear inverted pendulum system, the pendulum rod is stabilized in its upward direction by moving the attached cart. This is one of the benchmarks for research in control systems.

Inverted pendulum structure is a platform in which cart can only move in horizontal direction and pendulum rod (initially facing downwards) is attached with cart. Hence the rod is in its stable equilibrium state.

The purpose of this research paper is to analyze stability of inverted pendulum by varying either the rod's mass or rod's length or varying both simultaneously. Among the existing work, robustness analysis is done using fractional PID controller on ADAMS-MATLAB co-simulation using recursive least squares method in [1]. There is extensive literature on robustness testing of dynamic systems. But the method proposed is different in that it uses the concept of stability radius rather than usual concepts of gain and phase margins. In [2], linear matrix inequality (LMI) tool is used to test the robustness of system that is modeled using combination of Particle Swarm Optimization (PSO) and Takagi Sugeno (T-S) fuzzy control approach. In [3], analytical study of inverted pendulum system has been done using high frequency harmonic excitations. Larger the intensity of stochastic excitation or strength of frequency perturbations, larger is the Lyapunov exponent which destabilizes the system. Neural networks propose to behave more robust controller but in absence of disruptive effects over performances in perturbed conditions when balancing non-linear inverted pendulum system [4][5][6]. In [7], controller are designed to stabilize inverted pendulum in upright position using feedback by checking if sampling time delay was not large enough, controller keeps the system stabilized. Adaptive control schemes for inverted pendulum have been discussed in [8][9] where [8] presents the neural network based approach whereas sliding mode controller based approach has been explored in [9].

As the cart can move only in horizontal track and pendulum rod can move w.r.t a single angle, the dynamics of the system can be described as pendulum angle θ, pendulum angular velocity $\dot{\theta}$, card position x and cart velocity $\dot{x}$. This system also has some constraints like the cart can move up to a finite limited length and pendulum rod can move up to some finite limited angle on which the cart can move and steer back the pendulum rod to stabilize. Beyond that angle, the rod cannot be stabilized in its upright position. The pendulum rod in its upright position is in unstable equilibrium state when $\theta=0$ (vertically upward), some control method is needed to be applied to maintain the stability of the system.

Research has been done earlier on stabilization of inverted pendulum by using feedback controller response or by using fixed feedback. Stability of the system can also be analyzed with respect to variations in the mass of pendulum rod or in pendulum rod's length so that controller can behave more robust against external uncertainties. In this paper, we have made some variation in pendulum original mass and rod's length and analyzed the system stability. Major difference of the approach used in this paper from the existing work is that we have made use of the knowledge of stability radius for our analysis as opposed to conventional tools such as root locus Bode plot or Nyquist plot.

2. The Concept of Stability Radius:

The concept of stability radius is presented in [10]. Stability radius is defined as the radius of the smallest circle centered at the critical point ($-1 + 0j$) that touches the Nyquist plot of the open loop transfer function of the feedback control system. In order to clearly understand the definition, consider the feedback control system shown in Figure 1.

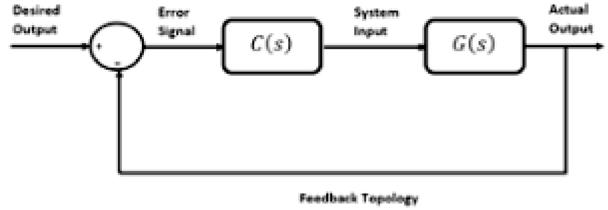

Figure 1: Feedback Control System

The loop transfer function for the above system is given by

$$L(s) = C(s)G(s) \tag{1}$$

The Nyquist plot of this loop transfer function is used to depict closed loop stability, gain margin and phase margin. The methods of finding the stability and the margins are in every control systems text book e.g. [11]. Stability radius is however a different concept as illustrated in Figure 2. As defined earlier, it is the radius of the circle touching the Nyquist plot of the open loop transfer function L/(s). In order to understand the importance of stability radius, one has to understand the importance of sensitivity function given by

$$S(s) = \frac{1}{1 + L(s)} \qquad (2)$$

The sensitivity function imposes many constraints on the closed loop performance [11]. Specifically, larger the value of sensitivity, more prone is the system performance to external disturbances. The way gain and phase margins are defined, it is possible to have a system with infinite gain margin and a good (> 60^0) phase margin and yet poor stability due to high value of sensitivity function.

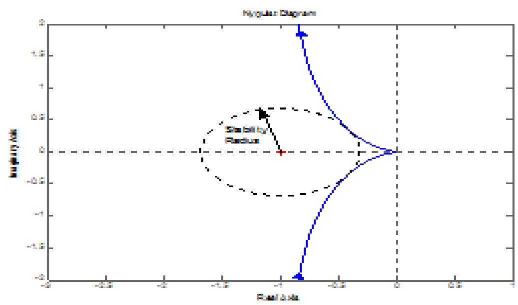

Figure 2: Example of Stability Radius

For example, in the system of Figure 2 *(L(s)=1/S²+s))* gain margin is infinite and phase margin is 51.8^0 yet sensitivity value is high for a range of frequencies. This tells us that it is important to analyze the stability radius of the systems. In this paper, we have done such analysis for an inverted pendulum because it is a benchmark system. Similar analysis can be performed on other systems in order to study the stability radius as a function of changes in system parameters. Such study can help in designing more robust and disturbance tolerant systems.

3. Inverted Pendulum Model

In figure 3, rod and cart system is shown on which force is being applied. Force on cart is represented as *N* in horizontal direction. Force on rod is represented as *P* in vertical direction.

From forces in horizontal direction (for the cart), the equation below can be obtained

$$M\ddot{x} = F - f\dot{x} - N \qquad (3)$$

where.

F = Force being applied on cart
x = Horizontal position of cart
f = damping coefficient
N = Force exerted on the cart in horizontal direction due to motion of the pendulum
M = mass of the cart

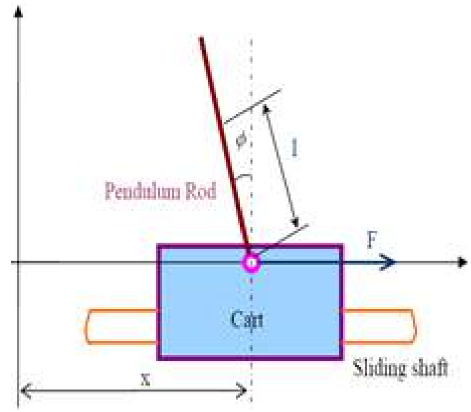

Figure 3: Inverted Pendulum on a Cart System

From the force acting on the rod in horizontal direction we get

$$N = m\frac{d^2}{dt^2}(x - lsin\phi) \qquad (4)$$

l=length of the pendulum rod
ϕ= Angle of the rod
m=mass of pendulum rod
Equation (4) can be written as,

$$N = m\ddot{x} - ml\ddot{\phi}cos\theta ml\dot{\phi}^2 sin\phi \qquad (5)$$

Substituting equation (5) into equation (3), first equation for non-linear system obtained is

$$(M + m)\ddot{x} + f\dot{x} - ml\ddot{\phi}cos\phi + ml\dot{\phi}^2 sin\phi = F$$

Similarly, combining the forces acting on the rod in vertical direction, we obtain second equation of motion

$$(I + ml^2)\ddot{\phi} - mglsin\phi = ml\ddot{x}cos\phi \qquad (6)$$

Linearization of equations (5) and (6) about the equilibrium point $[x \quad \dot{x} \quad \phi \quad \dot{\phi}]^T = [0 \quad 0 \quad 0 \quad 0]^T$ results in the following state space equation in matrix form.

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\phi}\\ \ddot{\phi}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \phi\\ \dot{\phi}\end{bmatrix} + \begin{bmatrix}0\\ \frac{I+ml^2}{I(M+m)+Mml^2}\\ 0\\ \frac{ml}{I(M+m)+Mml^2}\end{bmatrix}u \qquad (7)$$

where $u = F$ and the output equations can be written as,

$$y = \begin{bmatrix}x\\ \phi\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \phi\\ \dot{\phi}\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}u$$

By using equation (5), we can form

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\phi}\\ \ddot{\phi}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \frac{a_1}{b} & \frac{a_2}{b} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{a_3}{b} & \frac{a_4}{b} & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \phi\\ \dot{\phi}\end{bmatrix} + \begin{bmatrix}0\\ \frac{b_1}{b}\\ 0\\ \frac{b_1}{b}\end{bmatrix}u \qquad (8)$$

$$y = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \phi\\ \dot{\phi}\end{bmatrix}$$

$$SI - A = \begin{bmatrix}s & -1 & 0 & 0\\ 0 & s - \frac{a_1}{b} & \frac{-a_2}{b} & 0\\ 0 & 0 & s & -1\\ 0 & \frac{-a_3}{b} & \frac{-a_4}{b} & s\end{bmatrix} \qquad (9)$$

$$\det(SI - A) = s\left(\left(s - \frac{a_1}{b}\right)\left(\frac{-a_4}{b}\right) - \left(\frac{-a_2}{b}\right)\left(\frac{-a_3}{b}\right) + s\left(\left(s - \frac{a_1}{b}\right)s\right)\right)$$

$$\det(SI - A) = s\left(\frac{-(bs - a_1)(a_4)}{b^2} - \frac{a_2 a_3}{b^2} + s^2\left(\frac{bs - a_1}{b}\right)\right)$$

$$Adj(SI - A) = \begin{bmatrix}A_1 & A_2 & 0 & 0\\ 0 & A_3 & A_4 & 0\\ 0 & 0 & A_5 & A_6\\ 0 & A_7 & A_8 & A_9\end{bmatrix}^T$$

$$Adj(SI - A) = \begin{bmatrix}A_1 & 0 & 0 & 0\\ A_2 & A_3 & 0 & A_7\\ 0 & A_4 & A_5 & A_8\\ 0 & 0 & A_6 & A_9\end{bmatrix}$$

$$A_1 = \frac{1}{b}[(a_1 - bs)(a_4) - a_2 a_3 + s^2(bs - a_1)]$$

$$A_2 = 0$$

$$A_3 = \frac{s}{b}[bs^2 - a_4]$$

$$A_4 = \frac{a_2 s^2}{b}$$

$$A_5 = \frac{s^2}{b}(bs - a_1)$$

$$A_6 = \frac{s}{b}(bs - a_1)$$

$$A_7 = \frac{s^2 a_3}{b}$$

$$A_8 = \frac{1}{b^2}[a_4 s(bs - a_1) + a_2 a_3]$$

$$A_9 = \frac{s^2}{b}(bs - a_1)$$

Now, we have achieved the values for the system matrix for which we are going to analyze by changing length and mass to test the robustness of the system.

4. Stability Radius Analysis

In control systems, one of the dominant issue is to obtain robustness and stability. One of the main problem in analysis of robustness is to confirm that the system under study is able to maintain the stability under certain (defined/variable) conditions. Stability radius can be considered as the distance to instability. A system can be termed as natural robust if it has covered distance from point of instability to point of stability by keeping same dimensions. In different industry scenarios, it is conveniently easy to deal with polynomials of closed loop system e.g. for a system with single output or for with single inputs. One of the fundamental property of a system in closed loop analysis is to achieve or obtain the roots on a complex plane. If all the roots lie with stability region, the system (polynomial within complex variable) will be termed as stable.

All simulations are carried out using constant mass of the cart that is 0.792 kg. Increasing the length of the pendulum rod does not have much effect on stability radius as shown in Figure 4. The stability radius is changing in a little fraction with increase in pendulum rod's length as shown in Figure 5 (length variation from 0.304 m to 10 m in intervals of 0.8 m). Although the change in stability radius with respect to length is not very significant but reaching to a conclusion that increasing or decreasing the length does not affect stability would be too soon at this point. Therefore further analysis has been performed using the mass to length ratio in order to determine more realistic effect of change in length on the stability of the system. In practice, increasing the length does increase the overall mass of the system.

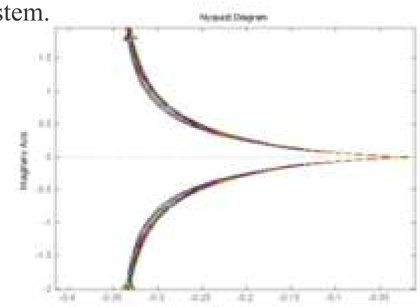

Figure4: Nyquist Diagram for variation of length

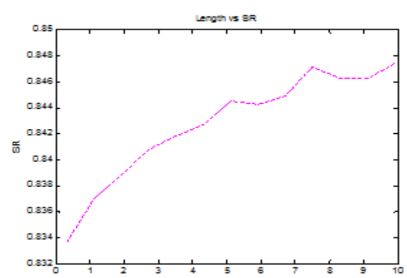

Figure5: Length vs Stability Radius

Now if we vary the mass on pendulum rod, the response and change in stability radius is decreasing. The response is shown in Figure 6 and Figure 7 (mass variation from 0.231 kg to 10 kg in intervals of 0.8 kg). Figure 6 shows that the Nyquist plot corresponding to various values of pendulum mass differ significantly. Comparing Figure 6 with Figure 4, the effect of variation in mass on stability is higher than that of length. But as stated earlier, variation in length does involve variation in mass. If we change both mass and length simultaneously, stability radius behavior shows abrupt decrease as shown in Figure 8 (mass over length ratio variation from 0.01 kg/m to 100 kg/m in intervals of 0.05 kg/m). This result indicate that the selection of mass to length ratio of inverted pendulum should be selected below 20 or 15 i.e. mass (in kilograms) should not be more than 20 times the length (in meters) for reasonable stability robustness. Such a conclusion is difficult to obtain from conventional root contour method and is very important for designing a robust stable system.

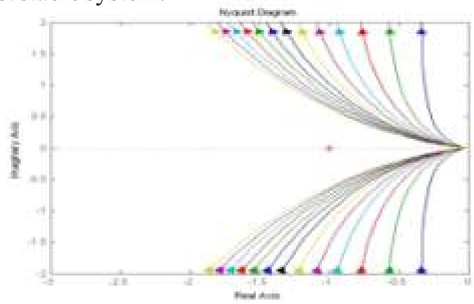

Figure6: Nyquist plot for variation of pendulum rod mass

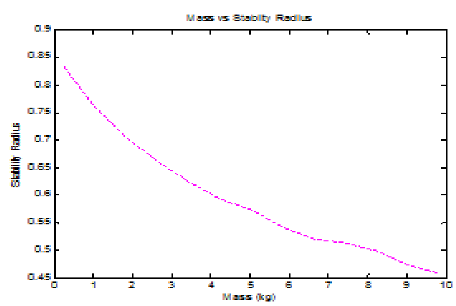

Figure7: Mass vs Stability Radius

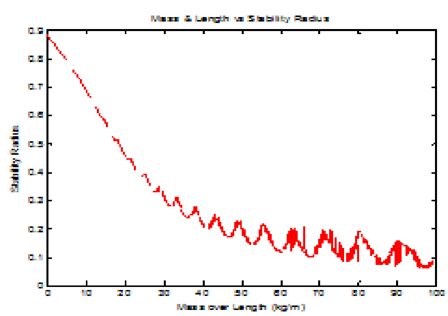

Figure8: Mass and length variation vs Stability Radius

1. Conclusions

Variations in the stability radius are shown for an inverted pendulum on a cart system as function of variations in mass and length of the rod. The results show that there is nontrivial variation in the stability radius when both parameters are changed simultaneously. It is also shown that the length alone does not affect the stability radius by a great amount. From these results, the motivation for the readers is to do such analysis on the systems to be controlled and identify parameters that affect the stability radius more than the others. Such analysis can help in achieving closed loop stability that is robust to parameter changes.

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