Investigation of Dynamical Systems with XPPAUT

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ABSTRACT

The mathematical modeling and computer simulations are extensively used in engineering problems and scientific research. The analysis of the system model gives us idea of understanding of the system whereas the simulation explains the feasibility whether system is implementable or not. Simulation also gives insight of the system realization. This paper focuses on interactive computational and simulation software XPPAUT (X-Windows Phase Plane Auto). Many Softwares i.e. MATLAB, MATHEMATICA, MAPLE are currently used for modeling and simulations of dynamical systems, but numerically integrating the differential equations is slower than that can be achieved with XPPAUT. Some problems incorporating the well-known linear and non-linear differential equations i-e radioactive decay, simple pendulum, Lorenz system, initial value problems and direction fields along-with null-clines has been coded, simulated and the results are presented to substantiate the claim.

Key Words: AUTO, Direction Field, Ordinary Differential Equations, Lorenz System, Simulation, XPPAUT.

1. INTRODUCTION

XPPAUT is freeware and it was developed by Ermentrout [1]. XPPAUT is an interactive package for differential equations, difference equations, delayed functions and boundary value problems. This computationally efficient analyzing tool can handle 590 differential equations at a time. The XPPAUT has two interactive and switchable parts XPP (X-Windows Phase Plane) and AUTO. It is an efficient and dedicated tool that is used for differential equations, difference equations, delayed function, initial value problems, stochastic problems and boundary value problems. XPP has 12 in-built ODEs (Ordinary Differential Equations) solvers that can integrate systems up to 590 differential equations [2]. It can pop-up 10 graphics windows in response to invoked inputs. It has user-friendly way of setting up the differential equations. It also supports equations with dirac-delta functions [1-3]. Nullclines, directions fields and phase planes can be plotted in much easier way as compared to any other numerical analysis tool. Just by clicking on nullclines, direction field options readily available in menu bar can plot nullclines and direction fields respectively.

The system requirements for installation are comparatively much lesser than any other technical and computational environments. The ODEs written is XPPAUT require less coding skills for the computing the response of the system under consideration. Many options are readily available as menu items of XPP main windows [2].

A prototype or model is an essential component for computer simulation. Fairly well representations of models are algebraic equations or differential equations [4-5]. Differential equation is a great tool for system modeling and simulation. The intrinsic behavior of the physical system can be predicted from the model and verified through the simulations. Differential equations are basically used to relate functions with their derivatives [5]. Thus differential equations help us understanding physical phenomena, laws of nature and system behaviors.

Dynamical systems are time dependent systems whose parameters or conditions may change with time [6]. These time-variant systems are modeled either as set of differential equations or difference equations. When systems are described by differential equations they are known as continuous dynamical systems and when these are modeled in terms of difference equations they are referred to as discrete dynamical systems. Both categories are analyzed and simulated for understanding and response of system through interactive simulating softwares. Analytical studies remains incomplete unless system is simulated and analyzed through any simulating tool so that system response and claim about system behavior can be justified [7].

2. COMPARISON WITH MATLAB, MAPLE AND MATHEMATICA

Scientists and researchers use MATLAB, MATHEMATICA, MAPLE to study and analyze the dynamical system [8-11]. These simulating packages are available for integration or simulation. MATLAB has flexibility regarding symbolic analysis and numerical simulations. It is decked with powerful parsing tool. In addition to this it has many built-in functions and libraries that enable it to solve problems related to dynamical systems. It comes with many example ODEs that are sufficient to explore, learn and familiarize one with the syntax, commands and functions. Its interface for writing up equations is very easy. Researchers with less programming knowledge can program or code equations very easily than that of any other simulating software. Likewise, in engineering education, circuit simulators are used to reinforce the student understanding of theoretical concepts using graphical aids [12-13].

XPP, which stands for X-windows phase space, is a graphical interface to AUTO. Qualitative analysis and numerical integration of dynamical systems is slower and effort requiring than that in XPPAUT.A big reason to use XPPAUT is that none of the other packages provide interface to AUTO facility [1]. AUTO is a package that allows you to see the response of dynamical systems as their initial conditions and parameters are varied [14-15]. AUTO is capable of finding the fixed points of the system as the parameter varies as well as tracking solutions to the differential equations.

3. SIMULATION RESULTS

Example-1: In this section, we will explore the radioactive decay model from the xppall/ode file that was downloaded with XPPAUT [2]. A simple program that is described by linear differential equations is discussed here. It has one parameter k. Then we set the initial conditions to execute the simulation. The linear ODE is a radioactive decay function that describes the decay of a radioactive element. The relation depicts that decay is independent of the initial amount of the substance. Decay just depends upon the half-life of the element.

The system is modeled as Equation (1).

$$\frac{dx}{dx} = -kx \quad (1)$$

Where x is the amount of substance available at any time t. The simulation code for linear ODE is as under:

(1) init x = 100
(2) par k = 0.054
(3) dx/dt= - kx
(4) @ dt= 0.025, total = 30, xplot = x, yplot = y, axes = 2d
(5) @ xmin = -30, x_{max} = 30, y_{min} = - 40, y_{max} = 40
(6) @ xlo = - 1.5, ylo = - 2, xhi = 1.5, yhi = 2
(7) done

The plot of radioactive decay of a certain substance is shown in Fig. 1. The curve is a decreasing function which shows decay of radioactive substance no matter with what initial condition we proceed. The negative sign in the ODE is an indication of decay of substance.

Example-2: In this section, we will now explore the Lorenz model from the xppall/ode file that was downloaded with XPPAUT [2]. Lorenz system is a non-linear, non-periodic three dimensional deterministic model for atmospheric convection. It consists of ODEs which are famous for their chaotic behavior due to change of parameters and initial conditions. Lorenz system is set of chaotic solution curves which upon plotting resembles a buttery or eight-curve.

The Lorenz system is described by following set of differential Equations (2-4).

$$\frac{dx}{dx} = s(-x + y) \quad (2)$$

$$\frac{dy}{dx} = rx - y - xz \quad (3)$$

$$\frac{dz}{dt} = - bz + xy \quad (4)$$

Where x, y and z describes the system state and b, r and s are system parameters.

The code is as follows:

(1) x = - 7.5 y = - 3.6 z = 30
(2) par r = 27 s = 10 b = 2.6
(3) dx/dt = s*(-x+y)
(4) dy/dt = r*x-y-x*z)
(5) dz/dt = - b*z + x *y
(6) @ dt = 0.025, total = 50, xplot = x, yplot = y, zplot = z, axes = 3d
(7) @ xmin = -30, xmax = 30, ymin = - 30, ymax = 30, zmin = 0, zmax = 40
(8) @ xlo = - 2.5, ylo = - 3, xhi = 2.5, yhi = 3
(9) @ maxstor = 25000
(10) @ phi = 60
(11) @ runnow = 1
(12) done

The plot is obtained for parameters as r =27,s=10=2.67.

We can see that two wings of the butterfly correspond to two different sets of physical behavior of the system as illustrated in Fig. 2. Any point on the plot represents a particular physical state, and curve shows the path followed by such a point during a finite period of time. Notice how the curve spirals around one wing a few moments before switching to the other wing.

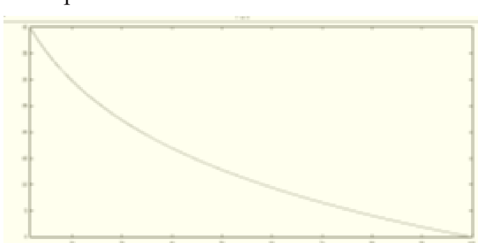

FIG. 1. RADIOACTIVE DECAY DESCRIBED BY LINEAR ODE

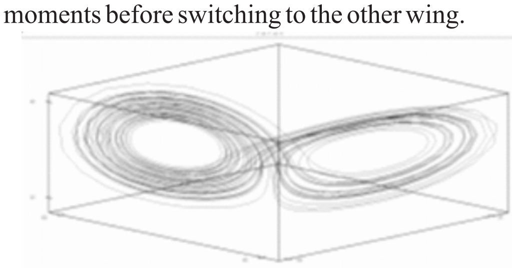

FIG. 2. 3D PLOT OF LORENZ SYSTEM

Example-3: In this section, we are going to explore simple pendulum model from the xppall/ode file that was downloaded with XPPAUT [2]. A simple pendulum consists of a bob of mass m suspended by light string of length l that is pivoted at some point P. When displaced to an initial angle and released, the pendulum will execute to and fro motion with some definite period. This system comprises of non-linear differential equations which are simulated with small angle approximations for understanding of physics of the system.

The system model is described by differential Equation (5).

$$\frac{d^2\theta}{dt^2} + \frac{g}{1}\sin\theta = 0 \tag{5}$$

The simulation code is as follows.

```
(1)  dx/dt = xp
(2)  dxp/dt = (- mu*xp- m*g*sin(x)/ (m*l))
(3)  PE = m*g*(1- cos(x))
(4)  KE = 0.5*m*l*xpaux^2
(5)  PE = PEaux
(6)  KE = KEaux
(7)  TE = PE + KE
(8)  param m = 15, mu = 2, g = 9.81, l = 1
(9)  param scale = 0.0083
(10) @ xp = x,yp = xp, xlo= - 5, xhi = 5, ylo = -10, yhi = 10
(11) @ bounds = 1000
(12) x(0) = 2
(13) done
```

The simulation is carried out and results are plotted with simulation tool. Plot describes that system response is periodic for given set of initial conditions. The simple pendulum while swinging has two forms of mechanical energies namely kinetic and potential. At extreme positions on either side it has max PE and at mean positions it has max KE. The system keeps on oscillating unless it loses all of its mechanical energy. The plot in the simulation consist of orbits that shows periodicity and oscillator orbit termination is an indication of the fact the system ultimately loses all of its energy. The plot is phase portrait which comprises of angle at which bob is displaced from mean position and angular velocity.

The system is periodic with period T as Equation (6).

$$T = 2\pi\sqrt{\frac{1}{g}} \tag{6}$$

Example-4: Now we will analyze and simulate a boundary value problem from the xppall/ode file that was downloaded with XPPAUT [2]. In the domain of ODEs, a BVP (Boundary Value Problem) is a pair of differential equations with some constraints. A solution of any BVP is solution to the differential equation which satisfies the given ODE and boundary conditions as well. We here discuss a non-linear BVP which consist of a differential equation and a set of boundary values. In XPPAUT we have "bdry" as keyword to specify boundary conditions [2].

We can simulate the problem with the following code.

```
(1)  du/dt = v
(2)  dv/dt = sin(t*u)
(3)  bdry u- 1
(4)  bdry du/dt
(5)  init u = 10,v = - 10
(6)  @ total = 20, bell = 0, xhi = 15
(7)  done
```

The BVP has been analyzed and simulated. Some initial and boundary constraints have been provided by using XPPAUT commands "init" and "bdry" [2]. The results are simulated for total time of 10 seconds as shown in Fig 3. The highest value of x is taken as 10, specified by "xhi" whereas "xlo" is by-default 0.

Example-5: Finally, we will analyze and simulate an exact ODE for direction field and null clines. In XPPAUT there are commands "Direction Field" and "Null Clines". Just by clicking on this button we can obtain direction fields and null clines [2]. To analyze the direction field and null clines following code is compiled and executed.

The general form of an exact ODE is as Equation (7).

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{7}$$

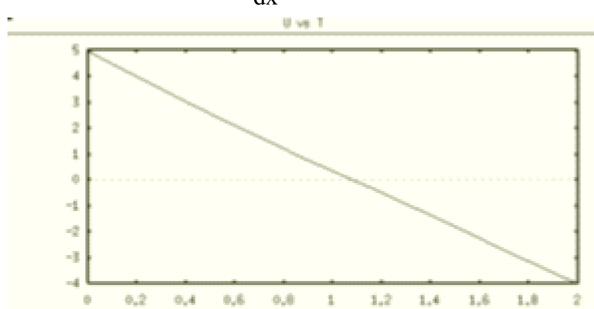

FIG. 3. A NON-LINEAR BOUNDARY VALUE PROBLEM

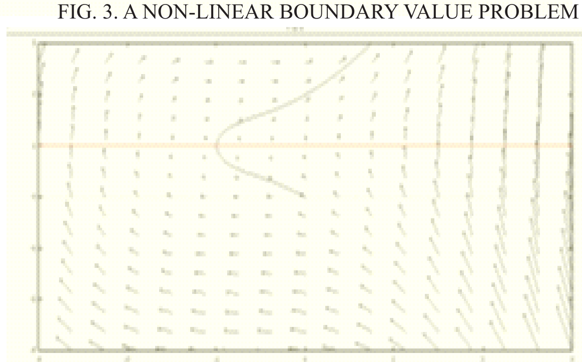

FIG. 4. DIRECTION FIELDS AND NULL CLINE

The equation selected for direction field is as Equation (8):

$$-(3x^2 + 4x + 2)\, dx + 2(y-1)dy = 0 \qquad (8)$$

(1) M(x,y) = - (3*x*x+4*x+2)
(2) N(x,y) = 2*(y- 1)
(3) dx/dt = N(x,y)
(4) dy/dt = - M(x,y)
(5) @ xp = x, yp = y, xlo = -5, xhi = 5, ylo = -5, yhi = 5
(6) done

In this section, qualitative attributes like direction fields and null clines are shown in Fig. 4. Null-cline is shown with red colored horizontal line while small lines with arrows indicate the direction field. Null cline is the curve where rate of change is zero. A solution curve is also plotted along with the said features.

4. DISCUSSION

The linear, non-linear differential equations and BVP were explored using XPPAUT. XPPAUT is a combination of XPP and AUTO [1-3]. It has been found great simulating software. It can interactively solve the given system of differential equations that means solution can be seen before the process termination, while MAPLE and MATHEMATICA are unable to do so [16-17]. AUTO is used for continuation and bifurcation of dynamical systems. In future its AUTO version will be explored.

AUTO option is available in XPP menu window that enable and prepare it for continuation and bifurcation analysis of the currently loaded problem. Continuation analysis describes the gradual development of solutions to differential equations over parameters, while bifurcation shows how solution curve appear and disappear as any of the system parameter is varied [1].

The qualitative methods i.e. phase portrait and directions fields give the general insight and interpretations of fluctuating dynamics of the system with perturbations in initial conditions and parametric variations [18-21]. The phase portrait in Fig. 5 describes the existence of any attractor, repeller and limit cycle with change of initial conditions and variations in parameters. In Fig. 5, we see limit cycle for different set of initial conditions.

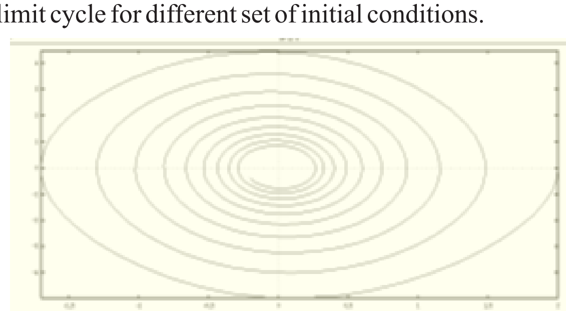

FIG. 5. SOLUTION CURVES OF SIMPLE PENDULUM

The direction field establishes a meaningful relation between the previous behaviors to the future response. The differential equation does not require analytical solution of the differential equations prior to generation of direction field. In Fig. 5 the small lines specify the flow direction to see the long term behavior of generic trajectories of an exact ODE [3].

5. CONCLUSION

The solution of ODEs has never been so easy. It requires analytical treatment of ODEs. Simulations need syntax familiarity and list of commands to execute a task. The main goal of the current study was to explore a simulator for dynamical systems. The mathematical treatment of differential equations with XPPAUT is found really straight forward. This software was explored, learned and found a powerful simulating tool for technical computing. Its user friendly, command oriented interface is easy to use. It does not require any proficiency in programming languages. The closed-form generic solutions are obtained with limited accuracy by employing methods named as direction field and isoclines. The same pair of differential equations has been simulated with said methods using MATLAB and XPPAUT. Null-Clines and Direction Filed has been plotted just by a click on null-cline and direction field options which are readily available on the main XPP window of XPPAUT. On contrast the plot of direction field in MATLAB needs a set of programming instructions i.e. "meshgrid" and "quiver" for every problem involving differential equations. System involving linear, non-linear ordinary differential equations and BVPs has been studied and solution curves have been discussed.

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